Integrity in Graphs: Bounds and Basics

Wayne Goddard[1]
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
USA

and

Henda C. Swart
Department of Mathematics,
University of Natal,
4001 Durban
Republic of South Africa

Abstract

The integrity of a graph G, denoted $I(G)$, is defined by $I(G) = \min\{\,|S| + m(G-S) : S \subset V(G)\,\}$ where $m(G-S)$ denotes the maximum order of a component of $G-S$; further an I-set of G is any set S for which the minimum is attained. Firstly some useful concepts are formalised and basic properties of integrity and I-sets identified. Then various bounds and interrelationships involving integrity and other well-known graphical parameters are considered, and another formulation introduced from which further bounds are derived. The paper concludes with several results on the integrity of circulants.

1 Introduction

Integrity was introduced by Barefoot, Entringer and Swart [3] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices. The motivation was that, in some respects, connectivity is oversensitive to local weaknesses and does not reflect the overall vulnerability. For example, the stars $K(1, n+1)$ and the graphs $K_1 + (K_1 \cup K_n)$ (where $+$ and $\cup$ denote the join and disjoint union) are all of connectivity one but differ vastly in how much damage is done to the corresponding communications network by the removal of a cut vertex: in the former case

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all communications are destroyed, whereas in the latter all but two stations remain in mutual contact.

It is our aim to lay the groundwork on integrity as a specific graphical parameter, especially with regard to the fundamental properties of I and I-sets and the interrelations among integrity and other graphical parameters. We also consider the integrity of some circulants; circulants are receiving a lot of attention in vulnerability studies (cf. for example [1], [2]).

We give first the requisite definitions and formalise some concepts which are used in the calculation of integrity, noting the integrity of some well-known (classes of) graphs. From there we explore the basic properties of integrity and of I-sets, consider various bounds and relationships involving integrity and other graphical parameters, and introduce another useful formulation. We conclude with some results on the values of the integrity of circulants.

2 Definitions

In this section, we define integrity and related concepts, and introduce the necessary terminology and notation. All undefined terminology and notation is taken from [5]. Specifically we use $p(G)$ to denote the *order* or number of vertices of a graph G, and $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degrees of G. Also, $\alpha(G)$, $\beta(G)$, $\chi(G)$, and $\omega(G)$ will denote the vertex cover, independence, chromatic and clique numbers respectively. Further a *cut-set* is any set of vertices whose removal leaves a disconnected graph while we shall use $\subset$ to denote strict containment.

The following definitions are from [3]:

- $\triangleright$ For any graph G, the maximum order of a component of G is denoted by $m(G)$.

- $\triangleright$ For any graph G the *integrity* of G, denoted $I(G)$, is defined by

$$I(G) = \min_{S \subset V(G)} \{\, |S| + m(G - S)\,\}. \tag{1}$$

- $\triangleright$ An *I-set* of G is any (strict) subset S of $V(G)$ for which $I(G) = |S| + m(G - S)$.

Two concepts that are useful computationally are now introduced:

▷ For any graph G,
$D_k(G) = \min\{\,|S| : S \subset V(G) \,\&\, m(G-S) \le k\,\} \quad k = 1, 2, \ldots$
$E_l(G) = \min\{\,m(G-S) : S \subset V(G) \,\&\, |S| = l\,\} \quad l = 0, 1, \ldots, p(G)-1.$

It is easily seen that the definition of $E_l(G)$ is unaffected by the replacement of the condition '$|S| = l$' by the inequality '$|S| \le l$'. It is also obvious that, for any graph G,

$$D_1(G) = \alpha(G) \qquad \text{and} \qquad E_0(G) = m(G).$$

Further, we have the following alternative formulations for integrity:

$$I(G) = \min_k (D_k(G) + k) \tag{2}$$

and

$$I(G) = \min_{0 \le l < p(G)} (E_l(G) + l). \tag{3}$$

We list the following known results for reference:

Proposition 1 *Let G be a graph of order n.*
a) $I(G) = n$ iff $G \cong K_n$.
b) $I(G) = 1$ iff $G \cong \bar{K}_n$.

Proposition 2 *[3]*
a) $I(K(a_1, a_2, \ldots, a_r)) = \sum_i a_i + 1 - \max_i a_i$.
b) $I(K(a, b)) = \min\{a, b\} + 1$.

Proposition 3 *[4]*
a) $I(P_n) = \left\lceil 2\sqrt{n+1} \right\rceil - 2 \quad n = 1, 2, \ldots$
b) $I(C_n) = \left\lceil 2\sqrt{n} \right\rceil \quad n = 3, 4, \ldots$

3 Basic Properties

In this section, we list simple but useful properties of I and I-sets.

The following result is easily proved:

Lemma 4 *For any graphs G and H,*
a) if $G \subseteq H$ then $I(G) \le I(H)$,
b) if G is non-trivial then for all $v \in V(G)$, $I(G - v) \ge I(G) - 1$,
c) for all $e \in E(G)$, $I(G - e) \ge I(G) - 1$.

Similar results may be formulated for D_k (and indeed the above follows from such observations). However while $E_l(G) \leq E_l(H)$ if $G \subseteq H$ (for a given $l < p(G)$), we note that the difference between $E_l(G)$ and $E_l(G-v)$, and that between $E_l(G)$ and $E_l(G-e)$, can both be made arbitrarily large. Consider, for example, $l = 0$ and the graph G_n formed by taking two disjoint copies of K_n and inserting a single edge e between them. Letting v be one of the endpoints of e, one gets that $E_0(G_n) = 2n$ while $E_0(G_n - e) = n = E_0(G_n - v)$.

We now list some properties of I-sets. The following is easily proved:

Lemma 5 *For all graphs G,*
a) if G is incomplete then every I-set of G is a cut-set of G and hence has cardinality at least $\kappa(G)$.
b) G is complete iff every strict subset of $V(G)$ is an I-set of G.

Lemma 6 *For all graphs G and H,*
a) if S is a minimal I-set of G, then for all $v \in S$, v is a cut-vertex of $G-(S-v)$,
b) if G has a complete component F, then there exists an I-set S of G such that S and V(F) are disjoint,
c) if G is a spanning subgraph of H and I(G)=I(H) then every I-set of H is an I-set of G,
d) if S is an I-set of G then $m(G-S) = I(G-S)$ and $\emptyset$ is an I-set of $G-S$,
e) if an I-set S of G exists such that $k(G-S) = k(G)$ then $\emptyset$ is an I-set of G.

Proof: a) We consider a minimal I-set, S, of G and *suppose* to the contrary that $v \in S$ and v is not a cut-vertex of $G-(S-v)$. Then $m(G-(S-v)) \leq m(G)+1$ so that $S-v$ is an I-set—a contradiction.
b) This follows from a) in that every minimal I-set has the required property.
c) Let S be an I-set of H. Then $I(G) \leq m(G-S)+|S| \leq m(H-S)+|S| = I(H) = I(G)$, proving the result.
d) Let T be an I-set of $G-S$. Then

$$\begin{aligned} |S| + m(G-S) &= I(G) \\ &\leq m(G-(S\cup T)) + |S \cup T| \\ &= |S| + m((G-S)-T) + |T| \\ &= |S| + I(G-S). \end{aligned}$$

Thus $I(G-S) \geq m(G-S)$ but $I(G-S) \leq m(G-S)$ and hence the result follows.
e) $G-S$ contains at least one vertex from each component of G (consider the return of one vertex) so that each component of G is represented by one component in $G-S$; specifically the largest component G has been trimmed but not disconnected. Thus $m(G-S) \geq m(G) - |S|$ so that $m(G-\emptyset) + |\emptyset| \leq m(G-S) + |S| = I(G)$ and the result is proved. □

One may also define integrity 'recursively':

Theorem 7 *For every nontrivial graph G,*

$$I(G) = \min \begin{cases} 1 + \min_{v \in V(G)} I(G-v), \\ m(G). \end{cases}$$

Proof: We note that $I(G) = m(G)$ iff $\emptyset$ is an I-set of G. Thus, if $I(G) < m(G)$ then

$$\begin{aligned} I(G) &= \min_{\emptyset \subset S \subset V(G)} \{|S| + m(G-S)\} \\ &= \min_{v \in V(G)} \min_{T \subset V(G-v)} \{|T \cup \{v\}| + m(G - (T \cup \{v\}))\} \\ &= \min_{v \in V(G)} \left\{ 1 + \min_{T \subset V(G-v)} \{|T| + m((G-v) - T)\} \right\} \\ &= 1 + \min_{v \in V(G)} I(G-v) \end{aligned}$$

proving the theorem. □

Using the above theorem and Lemma 5 we get:

Corollary 7
If G is connected and nontrivial then $I(G) = 1 + \min_{v \in V(G)} I(G-v)$.

4 Bounds

In this section we give some bounds involving I and other graphical parameters. The most obvious relationship is that $I(G) \geq \omega(G)$ which follows from Lemma 4. We follow with a useful theorem, the lower bound of which is also proved in [4]:

Theorem 8 *For every graph G, $\delta(G) + 1 \leq I(G) \leq \alpha(G) + 1$.*

Proof: To establish the lower bound, let S be an I-set of G. Then $m(G-S) \geq \delta(G-S)+1 \geq \delta(G)-|S|+1$ so that $I(G)=|S|+m(G-S) \geq \delta(G)+1$. To establish the upper bound, use the alternative formulation 2 noting that $D_1(G)=\alpha(G)$. □

Corollary 8 *If $\delta(G)=\alpha(G)$ then $I(G)=\delta(G)+1=\alpha(G)+1$.*

The corollary could be used to verify the integrity of the complete multipartite graphs (Proposition 2). We include another example of equality at the upper bound in the theorem which we shall use to illustrate the sharpness or otherwise of some of the bounds. Indeed this graph is nice for vulnerability consideration with its spread of degrees. Consider the class of graphs $\mathcal{B}$ defined by

$$\begin{aligned} B_1 &= K_2 \\ B_{j+1} &= (B_j \cup K_1) + K_1 \qquad j=1,2,\ldots \end{aligned} \tag{4}$$

(This may also be defined as the unique graph on $2j$ vertices with degree sequence $2j-1, 2j-2, \ldots, j+1, j,\ j, j-1, \ldots, 1$.) It is easily seen (by induction, say) that $\alpha(B_j)=j$ and $\omega(B_j)=j+1$ and hence by Theorem 8 and the opening observations,

$$I(B_j)=j+1 \tag{5}$$

so that we have equality in the aforementioned results.

We now proceed to investigate general equality at the bounds. Not too much can be said about equality at the upper bound though we shall have more to say about this in the next section.

Theorem 9 *For all graphs G, $I(G)=\kappa(G)+1$ iff $\kappa(G)=\alpha(G)$.*

Proof: The 'if' part follows directly from the above corollary. To prove the 'only if' part, let $I(G)=\kappa(G)+1$. Certainly, if G is complete then the statement is true; thus we may assume that G is noncomplete. Let S be an I-set of G. Then, by Lemma 5, $|S| \geq \kappa(G)$ and therefore $m(G-S)=1$ (i.e. S is a vertex cover of G) and $|S|=\kappa(G)$. Hence $\alpha(G) \leq |S|=\kappa(G)$ and the result follows. □

Theorems 8 and 9 show that one cannot arbitrarily prescribe κ, I and α. But these are the only restrictions, for we may construct a graph G with $\kappa(G)=k$, $I(G)=i$ and $\alpha(G)=a$ iff either $i=k+1=a+1$ or $k+1<i \leq a+1$. For example:

- $k+1 = i = a+1$:   Let $G = K_{k+1}$.
- $k+1 < i < a+1$:   Let $G = K_k + (K_{i-k} \cup (a+1-i)K_2)$.
- $k+1 < i = a+1$:   Let $G = K_k + (K_{i-k} \cup K_1)$.

One can also categorise graphs G such that $I(G) = \delta(G)+1$:

Theorem 10 *$I(G) = \delta(G)+1$ iff $G \cong F + nK_j$ where n, j are positive integers and F is a graph such that $\delta(F) \geq (p(F)-1)-(n-1)j$ or the null graph.*

Proof: Let G be a graph such that $i = I(G) = \delta(G)+1$. Let S be an I-set of G of cardinality s. Then

$$\delta(G-S)+1 \leq m(G-S) = i-s = \delta(G)+1-s \leq \delta(G-S)+1$$

yielding $\delta(G-S)+1 = m(G-S) = i-s$. This implies that $G-S \cong nK_j$ for some positive integer n, where $j = i-s$. If $S = \emptyset$ then we are done (Let F be the null graph). Assume therefore that $s > 0$. For any $u \in V(G-S)$, $\deg_G u \geq \delta(G) = i-1$ and $\deg_{G-S} u = i-s-1$ so that u is adjacent to every vertex in S. This yields that $G \cong \langle S\rangle + nK_j = F + nK_j$ (say). Further, for any vertex v in S, $\deg_G v = n(i-s)+\deg_F v \geq \delta(G) = i-1$; so that $\delta(F) \geq (s-1)-(n-1)j$. Conversely, the given conditions force $\delta(G) \geq p(F)+j-1$ while, considering $V(F)$ as a potential I-set yields $I(G) \leq p(F)+j$. □

As a further consideration, note that for every pair of positive integers i, p such that $p \geq i$, there exists a graph G of order p, integrity i and minimum degree $i-1$. Consider, for example, G to be $K_{i-1} + \bar{K}_{p+1-i}$ if $i \geq 2$ and $\bar{K}_p$ otherwise.

The next theorem is an extension of the bound $I(G) \geq \delta(G)+1$.

Theorem 11 *If G is any graph with degree sequence $d_1, d_2, \ldots, d_p$ where $d_1 \geq d_2 \geq \ldots \geq d_p$, then*

$$I(G) \geq \min_{1\leq t\leq p} \max\{t, d_t+1\}.$$

Proof: Consider any set $S \subset V(G)$ of cardinality s. Then $m(G-S) \geq 1$, obviously. Furthermore

$$m(G-S) \geq \Delta(G-S)+1 \geq \max_{v\in G-S} \deg_G v - s + 1 \geq d_{s+1} - s + 1.$$

Hence $|S| + m(G - S) \geq \max\{s + 1, d_{s+1} + 1\}$ and the result follows. □

We note, in passing, that the bound $\min_{1 \leq t \leq p} \max\{t, d_t + 1\}$ lies between $\delta(G) + 1$ and $\Delta(G) + 1$. Hence, we can find graphs for which the difference between the bound and the actual integrity is as large as we choose (for example C_n).

Nevertheless, we have the following sequence of inequalities:

$$\begin{aligned} I(G) &\geq \min_t \max\{t, d_t + 1\} \\ &\geq \max_t \min\{t, d_t + 1\} \\ &\geq \chi(G) \\ &\geq \omega(G), \end{aligned} \tag{6}$$

which yields a corollary to the theorem for which we also supply an independent proof:

Theorem 12 *For all graphs G, $I(G) \geq \chi(G)$.*

Proof: If G has chromatic number n, then G has an n-critical subgraph H; thus G has a subgraph H for which $\delta(H) \geq n - 1$ and consequently $I(G) \geq I(H) \geq \delta(H) + 1 \geq n = \chi(G)$. □

By the series of inequalities in 6, we have equality in Theorems 11 and 12 for the graphs of our class $\mathcal{B}$; further, equality is achieved in Theorem 11 but not in Theorem 12 (in general) by the complete multipartite graphs.

The next theorem provides an improvement of the lower bound $I(G) \geq \kappa(G) + 1$, bringing in a relationship amongst κ, β and I:

Theorem 13 *For all graphs G,*

$$I(G) \geq \left\lceil \frac{p(G) - \kappa(G)}{\beta(G)} \right\rceil + \kappa(G).$$

Proof: If G is complete then $I(G) = \text{RHS} = p$; so let us assume that G is not complete and let $S \subset V(G)$. Let $k(G)$ denote the number of components of a graph G. Then $k(G - S) \leq \beta(G - S) \leq \beta(G)$ so that

$$m(G - S) \geq \left\lceil \frac{p(G - S)}{k(G - S)} \right\rceil \geq \left\lceil \frac{p - |S|}{\beta(G)} \right\rceil.$$

Noting that every I-set S^* of G has order at least $\kappa(G)$, we obtain

$$I(G) = m(G - S^*) + |S^*| \geq \left\lceil \frac{p(G) - \kappa(G)}{\beta(G)} \right\rceil + \kappa(G)$$

which is the required result. □

The bound is sharp, equality being attained for, inter alia, graphs G with $\alpha(G) = \kappa(G)$ or $\alpha(G) = \kappa(G) + 1$. Nevertheless, we can find graphs G for which the difference between the bound and the actual value of $I(G)$ is as large as we please. Take for example the graphs B_j of our class $\mathcal{B}$, for $j \geq 2$:

$$I(B_j) = j + 1 \quad \text{while} \quad \left\lceil \frac{p(B_j) - \kappa(B_j)}{\beta(B_j)} \right\rceil + \kappa(B_j) = \left\lceil \frac{2j-1}{j} \right\rceil + 1 = 3.$$

5 An Alternative Formulation

In this section we introduce an auxiliary parameter which can simplify the calculation of integrity. We then derive further bounds using this formulation.

▷ For any graph G, $\theta(G) = p(G) - m(G)$.

Thus, $\theta(G)$ is the sum of the orders of all but a largest component of G. One can immediately write

$$I(G) = p(G) - \max_{H \prec G} \theta(H)$$

for, as S ranges through all strict subsets of $V(G)$, $H = G - S$ ranges through all induced subgraphs of G.

But, we can go further and restrict the subgraphs H we have to consider:

▷ $\mathcal{H}(G) = \{\, H \prec G : H \text{ contains two components of order } m(H) \,\}$

If $\theta(H) > 0$ then we can repeatedly remove non-cut-vertices from the largest component of H if necessary to yield a subgraph $H' \in \mathcal{H}(G)$ such that $\theta(H') = \theta(H)$. This yields on the convention $\max \emptyset = 0$:

$$I(G) = p(G) - \max_{H \in \mathcal{H}(G)} \theta(H) \tag{7}$$

This leads to a characterisation of integrity in terms of forbidden subgraphs, (cf. for instance, Theorem 15). We further note that if for any positive integer r there exists $H \in \mathcal{H}(G)$ with $\theta(H) \geq r$, then there exists $H' \in \mathcal{H}(G)$ with $\theta(H') = r$.

These concepts enable us to prove the following theorems.

Theorem 14 *If $2K_2 \not\prec G$ then $I(G) = \alpha(G) + 1$.*

Proof: By the definition of $\mathcal{H}$ and the preceding discussion, we see that if $2K_2 \not\prec G$ then the only elements of $\mathcal{H}(G)$ are those induced subgraphs H of G with $m(H) = 1$. Thus the vertex set of H is an independent set in G and so

$$\begin{aligned} I(G) &= p(G) - \max\{\, |S| - 1 : S \text{ is an independent set of } G \,\} \\ &= p(G) - (\beta(G) - 1) \\ &= \alpha(G) + 1 \end{aligned}$$

proving the result. □

Theorem 15
a) $I \leq p - 2$ iff $3K_1 \prec G$ or $2K_2 \prec G$.
b) $I = p - 1$ iff $\bar{G}$ is nonempty and has girth at least 5.

Proof: a) Using equation 7 and the subsequent discussion, we note that $I(G) \leq p - 2$ iff there exist induced subgraphs $H \in \mathcal{H}(G)$ with $\theta(H) = 2$. These are precisely $3K_1$ and $2K_2$.
b) $I \leq p - 1$ iff $\bar{G}$ is non-empty. Further

$$\begin{aligned} I \geq p - 1 \quad &\text{iff} \quad 3K_1 \not\prec G \text{ and } 2K_2 \not\prec G \\ &\text{iff} \quad K_3 \not\prec \bar{G} \text{ and } C_4 \not\prec \bar{G} \\ &\text{iff} \quad \bar{G} \text{ has girth at least } 5, \end{aligned}$$

where the last equivalence follows from noting that if C_4 is a subgraph of a graph but not induced, then the graph must contain a K_3. Combining these two results yields the theorem. □

Theorem 16 *For every graph G and nonnegative integer r, if $D_j(G) + j \geq p - r$ for $j = 1, 2, \ldots, r + 1$ then $I(G) \geq p - r$.*

Proof: Suppose to the contrary that $I(G) < p - r$. Then there exists an $H \in \mathcal{H}(G)$ such that $\theta(H) = r + 1$. Letting $m = m(H)$, we see that by the definition of $\mathcal{H}(G)$ it follows that $m \leq \theta(H) = r + 1$. But

$$\begin{aligned} D_m(G) + m &\leq |V(G) - V(H)| + m \\ &= p - (m + r + 1) + m \\ &< p - r, \end{aligned}$$

and a contradiction results. □

6 Circulants

The integrity of the most obvious circulants, namely the powers of the cycle was determined in [4]. There it was shown that:

Theorem 17 *[4] If* $1 \leq a \leq n/2$ *then* $I(C_n^a) = a(\psi - 1) + \lceil n/\psi \rceil$ *where* ψ *is given by* $\left\lceil \sqrt{n/a + 1/4} - 1/2 \right\rceil$.

In contrast to the arithmetical manipulation of the above proof, the value of the integrity of the complement of the powers of the cycle is relatively easily determined for some powers, as is shown in the following theorem, which is an extension of (part of) Theorem 15.

Theorem 18 *Consider* $G = C_n^a$ *where* $1 \leq a < n/4$. *Then* $I(\bar{G}) = n - a$.

Proof: Since $a < (n-1)/2$ it holds that $\omega(G) = a + 1$ and thus $\alpha(\bar{G}) = n - a - 1$. Further, since $4a < n$, G has no induced cycles of length four so that $2K_2 \not\prec \bar{G}$ and thus by Theorem 14 the result follows. □

This result may be extended:

Theorem 19 *Let* b *be a nonnegative integer. Define for* $n+b \equiv 0 \pmod 4$, $G_n = \bar{C}_n^a$ *where* $a = (n+b)/4$. *Then for sufficiently large* n, $I(G_n) = n - a$.

Proof: As in the above proof, $\alpha(G_n) = n - a - 1$ provided $2a < n - 1$. Now, for all $F \prec G_n$ such that $F \cong 2K_2$, define F' as $G_n - X$ where X is the set of all vertices of G_n adjacent to vertices from both components of F. Then let

$$\mathcal{M}_n = \left\{ F' : F \prec G_n \text{ and } F \cong 2K_2 \right\}.$$

Then for n sufficiently large, $\mathcal{M}_n$ contains, up to isomorphism, a fixed collection of graphs, say $\mathcal{G}_b$. Now, any induced subgraph H of G_n with two nontrivial components contains two edges which lie in different components of H and hence induce a $2K_2$. Further, any vertex which is adjacent to an end-vertex of both edges does not lie in H and thus every such H is an induced subgraph of some graph in $\mathcal{M}_n$. Hence the maximum value of $\theta(H)$, taken over all $H \in \mathcal{H}(G_n)$ such that $m(H) > 1$, can be determined from $\mathcal{M}_n$ and hence for n sufficiently large, from $\mathcal{G}_b$; it is, therefore, some value f_b, say. Thus $I(G_n) = \min\{n - a, n - f_b\} = I(G_n) = n - a$ for sufficiently large n and our result is proved. □

n	Det. set of G	$I(G)$	$I(\bar{G})$	Method
4	$\{1\}$	3	2	A
5	$\{1\}$	4	4	B
6	$\{1\}$	4	5	B–C
6	$\{2\}$	3	4	A
6	$\{3\}$	2	5	A
7	$\{1\}$	5	6	B–C
8	$\{1\}$	5	7	B–C
8	$\{2\}$	4	6	E
8	$\{4\}$	2	7	A
8	$\{1,2\}$	6	6	B–C
8	$\{1,3\}$	5	4	A
9	$\{1\}$	5	8	B–C
9	$\{3\}$	3	7	A
9	$\{1,2\}$	7	7	B–C
10	$\{1\}$	6	9	B–C
10	$\{2\}$	5	9	E–D
10	$\{5\}$	2	9	A
10	$\{1,2\}$	7	8	B–C
10	$\{1,3\}$	6	8	E
10	$\{1,5\}$	6	7	E
10	$\{2,4\}$	5	6	A
10	$\{4,5\}$	7	8	E–B

Table 1: The integrity of small-order circulants

For example, if $b = 0$ then for $n \geq 4$, $\mathcal{M}_n$ contains copies of $2K_2$ only while if $b = 1$ then for $n \geq 7$, $\mathcal{M}_n$ contains copies of P_6 only and thus $f_0 = f_1 = 2$. Thus we may extend Theorem 18 to include the case $a = n/4$ provided $a \geq 2$ i.e. $n \geq 8$, and the case $a = (n+1)/4$ provided ($a \geq 2$ and) $n \geq 7$.

We include a table of the integrity for small-order circulants (cf. table 1). Listed are the nonisomorphic circulants of orders at most ten for the non-empty noncomplete circulants. Various procedures may be employed to determine the relevant values. A suitable method is indicated by a letter or pair of letters in the 'method' column. These letters have the following meanings:

A: The graph or its complement is complete multipartite.

B: The graph is a cycle or power thereof so that the result follows from Proposition 3 or Theorem 17.

C: The graph is the complement of a cycle or power of cycle so that the result follows from Theorem 18 or its extension.

D: The graph is the complement of a graph of girth five so that the result follows from Theorem 15.

E: Some other method was used.

As an example of the other techniques that may be employed, consider the determination of the value of $I(G)$ for $G = C_{10}\langle 1, 3\rangle$. This (bipartite) graph has $D_1 = \alpha = 5$ (it is the complement of $K_2 \times K_5$) while $\kappa = 4$ so that $I(G) = 6$ by Theorem 9.

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