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ASYMPTOTIC BOUNDS ON THE INTEGRITY OF GRAPHS AND SEPARATOR THEOREMS FOR GRAPHS*

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Abstract. In this paper we study the integrity of certain graph families. These include planar graphs, graphs with a given genus, graphs on the d-dimensional integer lattice $\mathbb{Z}^d$, and graphs that have no K_h-minor. We give upper bounds for the integrity in terms of the order n of the graph. We also give lower bounds for box-graphs in $\mathbb{Z}^d$. As a consequence, the integrity of planar graphs is on the order of $n^{2/3}$, where $2/3$ is the best possible exponent.

Key words. integrity, planar graphs, lattice graphs, separators

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1. Introduction.

The *integrity* of a finite graph G is

$$I(G) = \min_{S \subset V(G)} \big(|S| + \tau(G \setminus S)\big),$$

where $\tau(G \setminus S)$ denotes the size of the largest component of $G \setminus S$. The integrity can be thought of as a measurement of connectivity of a graph. $|S|$ measures the amount of work needed to damage or disconnect a graph, while $\tau(G \setminus S)$ is a measure of how much of the graph is still intact. The integrity is the sum of these two quantities and was first introduced by Barefoot, Entringer, and Swart [4] inspired by the idea to measure a computer network's vulnerability.

Throughout this paper we assume that G is a graph with n vertices. It is easy to see that for the complete graph K_n, we have $I(K_n) = n$, and there are examples of simple, regular graphs with integrity of the order of n^α for any $0 \leq \alpha \leq 1$. However, the exact integrity of a given graph is difficult to compute. In fact, only for very simple graph families is the exact integrity known, so even establishing upper bounds for the integrity of large graph families is a worthwhile goal. See [3] and [8] for further information about the integrity of graphs.

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. An H*-minor* of G is a minor of G isomorphic to H. The *genus* g of a graph G is the smallest genus of all surfaces (compact orientable 2-manifolds) on which G can be properly embedded. In this paper we show that the integrity of graphs with no K_h-minor is $O(n^{2/3})$, where $h \geq 3$ is fixed. We give explicit upper bounds with particular attention to the case of planar graphs. The key property is that such graphs possess separator theorems of the form found in [1, 2, 5, 6, 7, 9]; see also section 2.

Our main results are Theorems 1.1, 1.2, 1.3, and 1.4 below.

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THEOREM 1.1. *Let G be a graph of order n with no K_h-minor for fixed $h \geq 3$. Then for $n \geq 119h^3$,*

$$I(G) \leq 10.9hn^{2/3} - 13.1h^{3/2}n^{1/2}.$$

THEOREM 1.2. *Let G be a planar graph of order n. Then for $n \geq 535$,*

$$I(G) \leq 18n^{2/3} - 27.9n^{1/2}.$$

THEOREM 1.3. *Let G be a graph of genus g and of order n. Then for $n \geq 713(2g+1)$,*

$$I(G) \leq 19.8(2g+1)^{1/3}n^{2/3} - 32.2(2g+1)^{1/2}n^{1/2}.$$

It follows that

$$I(G) = O\left(n^{2/3}\right)$$

for the graph families in Theorems 1.1, 1.2, and 1.3. Theorem 1.5 below shows that for planar graphs, $2/3$ is the best possible exponent.

Let $\mathbb{Z}^d$ denote the lattice graph where vertices are the points in $\mathbb{R}^n$ with integer coordinates, and vertices are adjacent if and only if their Euclidean distance is 1. A subgraph of $\mathbb{Z}^d$ which forms a rectangular box whose sides are parallel to the axes will be called a *box-graph*. The dimensions of a box-graph are the number of vertices lying on the edges of the box. (So, each dimension is the length of an edge plus 1.) The order of a box-graph is the product of its dimensions.

The theorem below provides a formula for calculating the integrity of a box-graph up to a constant factor depending on the dimension d only.

THEOREM 1.4. *Let G be a box-graph in $\mathbb{Z}^d$ with dimensions $a_1, \ldots, a_d$, where $a_1 \geq \cdots \geq a_d$ and set $m(G) = \sqrt[2]{a_1} + \sqrt[3]{a_1 a_2} + \cdots + \sqrt[d+1]{a_1 a_2 \ldots a_d}$. Then*

$$c_d \leq I(G) \bigg/ \left(\frac{|V(G)|}{m(G)}\right) \leq C_d, \tag{1.1}$$

where the constants c_d and C_d depend on d only.

Theorem 1.4 is equivalent to Lemma 4.2 in section 4 (the constants may differ). For planar box-graphs, the proof of Lemma 4.2 gives the following result.

THEOREM 1.5. *Let G be a planar box-graph of order n with dimensions a_1, a_2, and $a_1 \geq a_2$ (so $n = a_1 a_2$). If $a_2 \geq 2\sqrt{a_1}$, then*

$$0.00136n^{2/3} \leq I(G) \leq 5.22n^{2/3}.$$

If $a_2 < 2\sqrt{a_1}$, then

$$0.00136\sqrt{a_1}a_2 \leq I(G) \leq 5.22\sqrt{a_1}a_2.$$

Another special case of Theorem 1.4 is the following.

THEOREM 1.6. *Let G be the box-graph in $\mathbb{Z}^d$ which forms a cube. Let "a" denote the dimensions of the cube, so G has order $n = a^d$. Then there exist constants c_d and C_d depending on d alone such that*

$$c_d n^{\frac{d}{d+1}} \leq I(G) \leq C_d n^{\frac{d}{d+1}}.$$

Theorem 1.4 is also demonstrated in the following example for "flat" prism box-graphs.

Example 1.7. Let $G \subset \mathbb{Z}^3$ be a box-graph with dimensions a_1, a_2, a_3 and assume that $b := a_1 = a_2 \geq a_3 =: a$. In the notation of Lemma 4.2, $A_0 = 1$, $A_1 = \sqrt{b}$, $A_2 = \sqrt[3]{b^2}$, and $A_3 = \sqrt[4]{b^2 a}$. If $b \geq 4$ and $a < 2\sqrt[3]{b^2}$, then, in Lemma 4.2, we have $N = 2$. Now $|V(G)|/A_2 = ab^2/\sqrt[3]{b^2} = ab^{4/3}$, and so $c_d^* ab^{4/3} \leq I(G) \leq C_d^* ab^{4/3}$.

2. Separator theorems. For $A \subset V(G)$, we denote by $G[A]$ the induced subgraph of G. (This is a graph whose vertex set is A and where two vertices in $G[A]$ are connected by an edge if and only if they are connected by an edge in the graph G.)

Proposition 2.1. *Let $0 < \alpha \leq 1$ and $1 \leq c$. Let $\mathcal{G}_\alpha(c)$ be a family of graphs so that for any $G \in \mathcal{G}_\alpha(c)$, there exists a vertex partition $V(G) = A \cup B \cup C$, where $|A|, |B| \leq (2/3)|V(G)|$, $|C| \leq c|V(G)|^\alpha$, and no vertex in A is adjacent to a vertex in B. Suppose further that if $G \in \mathcal{G}_\alpha(c)$, then every subgraph of G is in $\mathcal{G}_\alpha(c)$. Then any $G \in \mathcal{G}_\alpha(c)$ can be partitioned*

$$V(G) = A' \cup B' \cup C',$$

where $|A'|, |B'| \leq (1/2)|V(G)|$,

$$|C'| \leq \frac{c}{1-(2/3)^\alpha}|V(G)|^\alpha,$$

and no vertex in A' is adjacent to a vertex in B'.

Proof. We follow the proof of Corollary 3 of [9]. We inductively define a sequence of sets $\{A_i, B_i, C_i, D_i\}$ so that $V(G) = A_i \cup B_i \cup C_i \cup D_i$ is a vertex partition; there are no edges between any of the sets A_i, B_i, and D_i, and we have $|A_i| \leq |B_i| \leq |A_i \cup C_i \cup D_i|$ and $|D_i| \leq \frac{2}{3}|D_{i-1}|$.

Let $A_0 = B_0 = C_0 = \emptyset$ and $D_0 = V(G)$. The properties above are clearly satisfied for these sets. Assume that $A_{i-1}, B_{i-1}, C_{i-1}, D_{i-1}$ have been defined satisfying the above properties and further assume that $D_{i-1} \neq \emptyset$. Applying the hypotheses on $\mathcal{G}_\alpha(c)$ to the graph $G[D_{i-1}]$, we have $D_{i-1} = \widetilde{A} \cup \widetilde{B} \cup \widetilde{C}$, where $|\widetilde{A}|, |\widetilde{B}| \leq \frac{2}{3}|D_{i-1}|$, $|\widetilde{C}| \leq c|D_{i-1}|^\alpha$, and no vertex in $\widetilde{A}$ is adjacent to a vertex in $\widetilde{B}$. We can assume that $|\widetilde{A}| \leq |\widetilde{B}|$.

Let A_i be the smaller (in cardinality) of the sets $A_{i-1} \cup \widetilde{A}$ and B_{i-1}. Let B_i be the other set. Let $C_i = C_{i-1} \cup \widetilde{C}$, and let $D_i = \widetilde{B}$. Then

$$\begin{aligned} A_i \cup B_i \cup C_i \cup D_i &= A_{i-1} \cup \widetilde{A} \cup B_{i-1} \cup C_{i-1} \cup \widetilde{C} \cup \widetilde{B} \\ &= A_{i-1} \cup B_{i-1} \cup C_{i-1} \cup D_{i-1} \\ &= V(G). \end{aligned}$$

Since no vertex in A_{i-1} is adjacent to one in B_{i-1} or $\widetilde{B}$, then no vertex in A_i is adjacent to one in B_i or D_i. Similarly, no vertex in B_i is adjacent to one in D_i.

Also, $|A_i| \leq |B_i|$ by our choice of A_i, and if $A_i = B_{i-1}$, then $B_i = A_{i-1} \cup \widetilde{A}$ and

$$\begin{aligned} |A_i \cup C_i \cup D_i| &\geq \left|B_{i-1} \cup \widetilde{B}\right| \\ &\geq \left|A_{i-1} \cup \widetilde{A}\right| = |B_i|. \end{aligned}$$

If $A_i = A_{i-1} \cup \widetilde{A}$, then $B_i = B_{i-1}$ and

$$\begin{aligned} |A_i \cup C_i \cup D_i| &\geq |A_{i-1} \cup C_{i-1} \cup D_{i-1}| \\ &\geq |B_{i-1}| = |B_i|. \end{aligned}$$

Also, $|D_i| = |\widetilde{B}| \leq \frac{2}{3}|D_{i-1}|$ by the hypotheses. It follows that each term in this sequence of subsets of $V(G)$ satisfies all of the above properties.

As the vertex set of G is finite and $|D_i|$ is decreasing, then $|D_k| = 0$ for some k. Thus for such k, we have $V(G) = A_k \cup B_k \cup C_k$, $|A_k| \leq |B_k| \leq |A_k \cup C_k|$, and no vertex in A_k is adjacent to one in B_k. Let $A' = A_k$, $B' = B_k$, and $C' = C_k$. It follows that $|A'|, |B'| \leq n/2$. Now,

$$\begin{aligned} |C_i| &= |C_{i-1}| + \left|\widetilde{C}\right| \\ &\leq |C_{i-1}| + c|D_{i-1}|^{\alpha}. \end{aligned}$$

As $|D_0| = n$, then $|D_i| \leq (2/3)^i n$, so

$$|C_i| \leq |C_{i-1}| + c(2/3)^{(i-1)\alpha} n^{\alpha}.$$

As $|C_i| \leq c(2/3)^{(1-1)\alpha} n^{\alpha} + c(2/3)^{(2-1)\alpha} n^{\alpha} + \cdots + c(2/3)^{(i-1)\alpha} n^{\alpha}$, then

$$\begin{aligned} |C'| &\leq \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{i\alpha} cn^{\alpha} \\ &= \frac{c}{1-(2/3)^{\alpha}}\, n^{\alpha}. \qquad \square \end{aligned}$$

Theorem 2.2 (Alon, Seymour, Thomas (1990) [1]). *Let G be a graph with n vertices and no K_h-minor, for fixed $h \geq 3$. Then there exists a partition $V(G) = A \cup B \cup C$ such that $|A|, |B| \leq 2n/3$, $|C| \leq h^{3/2} n^{1/2}$, and no vertex in A is adjacent to a vertex in B.*

A straightforward application of Proposition 2.1 and Theorem 2.2 gives the following.

Corollary 2.3. *Let G be a graph with n vertices and no K_h-minor, for fixed $h \geq 3$. Then $V(G) = A \cup B \cup C$, where $|A|, |B| \leq n/2$, $|C| \leq \frac{h^{3/2}}{1-\sqrt{2/3}}\, n^{1/2}$, and no vertex in A is adjacent to a vertex in B.*

The well-known separation theorem for planar graphs [9] was improved in [2] to give the best known such result thus far. See also [6] for results on the decomposition of planar graphs.

Theorem 2.4 (Alon, Seymour, Thomas (1994) [2]). *Let G be a planar graph with n vertices. Then there exists a partition $V(G) = A \cup B \cup C$ such that $|A|, |B| \leq 2n/3$, $|C| \leq 3\sqrt{2}/2\; n^{1/2}$, and no vertex in A is adjacent to a vertex in B.*

A straightforward application of Proposition 2.1 and Theorem 2.4 gives the following.

Corollary 2.5. *Let G be a planar graph with n vertices. Then there exists a partition $V(G) = A \cup B \cup C$ such that $|A|, |B| \leq n/2$, $|C| \leq \frac{3\sqrt{2}}{2(1-\sqrt{2/3})}\, n^{1/2}$, and no vertex in A is adjacent to a vertex in B.*

The separation theorem for planar graphs in [9] was generalized in [7, 5] to graphs with a fixed genus g. Below is the separator theorem from [5], which is slightly stronger than the theorem in [7].

Theorem 2.6 (Djidjev (1985) [5]). *Let G be a graph with n vertices and genus g. Then there exists a partition $V(G) = A \cup B \cup C$ such that $|A|, |B| \leq 2n/3$, $|C| \leq \sqrt{6(2g+1)n}$, and no vertex in A is adjacent to a vertex in B.*

A straightforward application of Proposition 2.1 and Theorem 2.6, together with the observation that a subgraph of a graph of genus g has a genus $\leq g$, gives the following.

Corollary 2.7. *Let G be a graph with n vertices and genus g. Then there exists a partition $V(G) = A \cup B \cup C$ such that $|A|, |B| \leq n/2$, $|C| \leq \frac{\sqrt{6(2g+1)}}{1-\sqrt{2/3}}\, n^{1/2}$, and no vertex in A is adjacent to a vertex in B.*

3. Upper bounds on the integrity of graphs. In this section we give upper bounds for the integrity of certain graphs, which have a separator theorem of the type given in Corollaries 2.3, 2.5, and 2.7.

THEOREM 3.1. *Let $0 \le \alpha < 1$ and $1 \le c$. Let $\mathcal{G}_\alpha(c)$ be a family of graphs so that for any $G \in \mathcal{G}_\alpha(c)$, with $|V(G)| = n$, there exists a partition $V(G) = A \cup B \cup C$ such that $|A|, |B| \le n/2$, $|C| \le cn^\alpha$, and no vertex in A is adjacent to a vertex in B. Further, assume that if $G \in \mathcal{G}_\alpha(c)$, then every subgraph of G is in $\mathcal{G}_\alpha(c)$. Then for any $G \in \mathcal{G}_\alpha(c)$ and $n \ge (2c)^{1/(1-\alpha)}$, we have*

$$I(G) \le an^{\frac{1}{2-\alpha}} - bn^\alpha,$$

where

$$a = c^{\frac{1}{2-\alpha}} 2^{-\frac{1-\alpha}{2-\alpha}} \left(1 + \frac{1}{1 - 2^{-(1-\alpha)}}\right)$$

and

$$b = \frac{c}{2^{1-\alpha} - 1}.$$

Note that $\frac{1}{2-\alpha} > \alpha$ for $0 \le \alpha < 1$.

Proof. Let $G \in \mathcal{G}_\alpha(c)$, with $|V(G)| = n$. Then $V(G) = A \cup B \cup C$ by the hypothesis. By removing the set of vertices C, we divide G into components $G[A]$ and $G[B]$, each of which has no more than $n/2$ vertices. This directly gives the estimate

$$I(G) \le \frac{n}{2} + cn^\alpha.$$

Now we apply the separator theorem to each of the subgraphs $G[A]$ and $G[B]$. Thus

$$A = A_1 \cup B_1 \cup C_1,$$

where $|A_1|, |B_1| \le n/4$, and $|C_1| \le c(n/2)^\alpha$ and similarly for $B = A_2 \cup B_2 \cup C_2$. By removing the vertices in C_1 and C_2, we decompose G into 4 components, each with no more than $n/4$ vertices. It follows that

$$\begin{aligned} I(G) &\le \frac{n}{4} + |C| + |C_1| + |C_2| \\ &\le \frac{n}{4} + cn^\alpha + 2c\left(\frac{n}{2}\right)^\alpha. \end{aligned}$$

Continuing in this way, we apply the separator theorem successively ℓ times (where ℓ is a nonnegative integer to be specified later). At each step, we remove vertices to separate each of 2^i components already obtained with a set of vertices of size no more than $c(n/2^i)^\alpha$. After i steps, we have decomposed G into 2^i components, each containing no more than $n/2^i$ vertices. At the ith-step we would remove no more than $2^{i-1}c(n/2^{i-1})^\alpha$ vertices.

It follows that for any nonnegative integer ℓ, we have the estimate

$$I(G) \le \frac{n}{2^\ell} + \sum_{i=0}^{\ell-1} c2^i \left(\frac{n}{2^i}\right)^\alpha \tag{3.1}$$

$$= n + \sum_{i=0}^{\ell-1} \left(c2^i \left(\frac{n}{2^i}\right)^\alpha - \frac{n}{2^{i+1}}\right). \tag{3.2}$$

Now we set a value for ℓ. Define

$$\ell = \max\left(0, 1 + \left[\log_2\left(\frac{n^{1-\alpha}}{2c}\right)^{\frac{1}{2-\alpha}}\right]\right).$$

(This value of ℓ minimizes (3.2). This follows from the fact that

$$c2^i\left(\frac{n}{2^i}\right)^\alpha - \frac{n}{2^{i+1}} \leq 0$$

if and only if $i \leq \log_2(\frac{n^{1-\alpha}}{2c})^{1/(2-\alpha)}$.)

Since $n \geq (2c)^{1/(1-\alpha)}$ implies $1 \leq n^{1-\alpha}/(2c)$, we get

$$\ell = 1 + \left[\log_2\left(\frac{n^{1-\alpha}}{2c}\right)^{\frac{1}{2-\alpha}}\right] \tag{3.3}$$

$$= 1 + \log_2\left(\frac{n^{1-\alpha}}{2c}\right)^{\frac{1}{2-\alpha}} - \log_2(1/\delta), \tag{3.4}$$

where $\log_2(1/\delta) \in [0,1)$ is the fractional part of the second term in (3.3). Thus

$$2^\ell = 2\left(\frac{n^{1-\alpha}}{2c}\right)^{\frac{1}{2-\alpha}}\delta \quad \text{for some} \quad \delta \in \left(\frac{1}{2}, 1\right].$$

Substituting this expression into the right-hand side of the estimate

$$I(G) \leq \frac{n}{2^\ell} + \sum_{i=0}^{\ell-1} c2^i\left(\frac{n}{2^i}\right)^\alpha = \frac{n}{2^\ell} + cn^\alpha\frac{(2^{1-\alpha})^\ell - 1}{2^{1-\alpha} - 1},$$

we get

$$I(G) \leq (cn)^{\frac{1}{2-\alpha}}2^{-\frac{1-\alpha}{2-\alpha}}\delta^{-1} + \frac{(cn)^{\frac{1}{2-\alpha}}2^{\frac{(1-\alpha)^2}{2-\alpha}}}{2^{1-\alpha}-1}\delta^{1-\alpha} - \frac{c}{2^{1-\alpha}-1}n^\alpha.$$

Let $f(\delta)$ denote the right-hand side of the above inequality. Straightforward calculations show that $\lim_{\delta\to 0^+} f(\delta) = \lim_{\delta\to+\infty} f(\delta) = +\infty$, and the only critical point of $f(\delta)$ on $(0, +\infty)$ is

$$\delta = \left(\frac{1 - 2^{-(1-\alpha)}}{1-\alpha}\right)^{\frac{1}{2-\alpha}}.$$

Furthermore, this critical point is in $(1/2, 1]$ for any $\alpha \in [0, 1)$. It follows that

$$\sup_{\delta\in(\frac{1}{2},1]} f(\delta) = \max\left(f\left(\frac{1}{2}\right), f(1)\right).$$

It is easy to verify that, in fact, $f(1/2) = f(1)$. Hence

$$I(G) \leq f(1) = (cn)^{\frac{1}{2-\alpha}}2^{-\frac{1-\alpha}{2-\alpha}}\left(1 + \frac{1}{1-2^{-(1-\alpha)}}\right) - \frac{c}{2^{1-\alpha}-1}n^\alpha. \qquad \square$$

Example 3.2. Let G be a graph which is the union of finitely many paths, and let $n = |V(G)|$. Theorem 3.1 now implies (with $\alpha = 0$ and $c = 1$) that $I(G) \leq \frac{3}{2}\sqrt{2n} - 1$. Note that this bound is quite sharp, as for the path P_n of length n, we have $I(P_n) = \lceil 2\sqrt{n+1}\rceil - 2$ and $\frac{3}{2}\sqrt{2} \approx 2.121$. As paths have the maximum integrity of trees of order n, see Lemma 5 from [10], we get $I(G) \leq \frac{3}{2}\sqrt{2n} - 1$ for G a tree.

Proof of Theorem 1.1. Let G be a graph with n vertices and no K_h-minor. Set $c = \frac{h^{3/2}}{1-\sqrt{2/3}}$ and $\alpha = 0.5$. Using Corollary 2.3 and Theorem 3.1, we get that for $n \geq 119h^3$, we have

$$I(G) \leq 10.9hn^{2/3} - 13.1h^{3/2}n^{1/2}. \qquad \square$$

Proof of Theorem 1.2. Let G be a planar graph with n vertices. Set $c = \frac{3\sqrt{2}}{2(1-\sqrt{2/3})}$ and $\alpha = 0.5$. Using Corollary 2.5 and Theorem 3.1 we get that for $n \geq 535$, we have

$$I(G) \leq 18n^{2/3} - 27.9n^{1/2}. \qquad \square$$

Proof of Theorem 1.3. Let G be a graph with n vertices and genus at most g. Define $c = \frac{\sqrt{6(2g+1)}}{1-\sqrt{2/3}}$ and $\alpha = 0.5$. Using Corollary 2.7 and Theorem 3.1, we gain that for $n \geq 713(2g+1)$, we have

$$I(G) \leq 19.8(2g+1)^{1/3}n^{2/3} - 32.2(2g+1)^{1/2}n^{1/2}. \qquad \square$$

4. Rectangular boxes in the lattice graph $\mathbb{Z}^d$. Let d be a positive integer. Recall that a subgraph G of $\mathbb{Z}^d$, which forms a rectangular box that is parallel to the axes, is called a box-graph. We say that G has dimensions $a_1, \dots, a_d$ if $a_1 \geq \cdots \geq a_d$, $a_i \in \mathbb{N}$, and G contains all vertices $(x_1, \dots, x_d)$, where $x_i \in \mathbb{Z}$ and $0 \leq x_i < a_i$ for all i. Let m_d denote the number of vertices on a smallest hyperface of a box-graph G. Then $m_d = \Pi_{i=2}^d a_i$. In the case of $d = 1$, we define $m_1 = 1$.

The following lemma asserts the (seemingly obvious) statement that if we delete a set of vertices S from a box-graph G and $|S|$ is small, then $G \setminus S$ will contain a large component.

LEMMA 4.1. *Let G be a box-graph in $\mathbb{Z}^d$. For any $\epsilon \in (0,1)$, there exists $c_d \in (0,1)$ such that if $S \subset G$ and $|S| \leq c_d m_d$, then there exists a component K of $G \setminus S$ such that $|K| \geq (1-\epsilon)|V(G)|$.*

Proof. The proof is by induction on the dimension d. We observe that the lemma is true for $d = 1$, since $|S| \leq c_1 m_1$, with $c_1 \in (0,1)$ implies $S = \emptyset$.

Let $d \geq 2$ and assume that our statement is true for $d-1$. Let c_d be a small number (which we specify later). Let a_1 be the largest dimension of the box-graph G, and let e be an "edge" of the box consisting of a_1 vertices lying on a line. (Here the word "edge" is used as in geometry, not in graph theory.) Set $u = a_1$.

Consider the u cross sections of G which are orthogonal to e. Observe that, by the choice of e, each such cross section consists of m_d vertices. Denote by $\mathcal{H}$ the set of those cross sections which have at most $\sqrt{c_d}m_d/u$ vertices from S. Note that $|S| \leq c_d m_d$ implies that there are less than $\sqrt{c_d}u$ cross sections not in $\mathcal{H}$, and therefore $|\mathcal{H}| > (1-\sqrt{c_d})u$. Let $R \in \mathcal{H}$. We regard R as a $(d-1)$ dimensional box-graph with $|R| = m_d$ vertices. Let m_{d-1} denote the minimal number of vertices on any of the $(d-2)$ dimensional faces of R.

Let $C \in (0,1)$, which we specify later. Note that, by the definition of u, we have $m_d/u \leq m_{d-1}$ (which is true even for $d = 2$), thus $|S \cap R| \leq \sqrt{c_d}m_{d-1}$. By

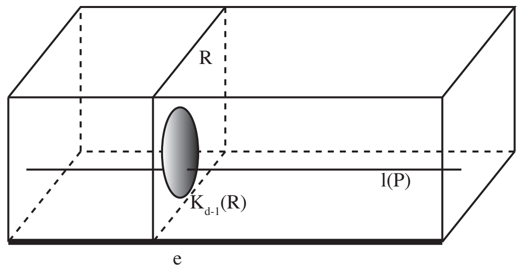

FIG. 4.1. *A schematic representation of a box-graph, with a cross section R and the component $K_{d-1}(R)$. Lines of the type $l(P)$ connect many such components from different cross sections.*

the induction hypotheses, if c_d is small enough, then *for any* $R \in \mathcal{H}$, there exists a component $K_{d-1}(R)$ of $R \setminus S$ such that

$$|K_{d-1}(R)| \geq Cm_d. \tag{4.1}$$

For an illustration, see Figure 4.1. Our goal is to construct the desired component K of $G \setminus S$ as a union of certain components $K_{d-1}(R)$ to yield the desired size.

For a vertex $P \in V(G)$, let $l(P)$ denote those vertices of G which are on the line passing through P and parallel to e. Let

$$G' := \{P \in V(G) \mid l(P) \cap S = \emptyset\}$$

and $T' := T \cap G'$, where T is an arbitrary fixed cross section of G orthogonal to e. Let

$$J := \cup_{R \in \mathcal{H}} (K_{d-1}(R) \cap G').$$

Since

$$|K_{d-1}(R) \cap G'| \geq |K_{d-1}(R)| - |S| \geq (C - c_d)m_d,$$

we have $|J| > (1 - \sqrt{c_d})u(C - c_d)m_d$. J may not be a connected subset of $G \setminus S$. To obtain a connected subset of J, we proceed as follows: $|T'| \leq m_d$ implies that there exists $P \in T'$ such that $|l(P) \cap J| > (1 - \sqrt{c_d})u(C - c_d)$. Thus the line segment $l(P)$ connects more than $(1 - \sqrt{c_d})u(C - c_d)$ of the sets $K_{d-1}(R)$ $(R \in \mathcal{H})$. Hence, there exists a component K of $G \setminus S$ such that

$$|K| > (1 - \sqrt{c_d})u(C - c_d)Cm_d = (1 - \sqrt{c_d})(C - c_d)C|V(G)|.$$

To complete the proof, we choose $C \in (0, 1)$ to be so close to 1 and then c_d to be so close to 0 such that (4.1) holds for any $R \in \mathcal{H}$ and

$$(1 - \sqrt{c_d})(C - c_d)C \geq 1 - \epsilon. \qquad \square$$

Proof of Theorem 1.4. We show that Theorem 1.4 follows from the following lemma.

LEMMA 4.2. *Let G be a box-graph in $\mathbb{Z}^d$, with dimensions $a_1, \ldots, a_d$, where $a_1 \geq \cdots \geq a_d$. Let $A_0 = 1$ and $A_m = (a_1 \ldots a_m)^{1/(m+1)}$ for $m \in \{1, \ldots, d\}$, and let*

$$\mathcal{N} := \left\{ m \in \{0, \ldots, d-1\} \;\middle|\; a_{m+1} < 2A_m \right\}.$$

If $\mathcal{N}$ is nonempty, then let $N := \min \mathcal{N}$, otherwise let $N := d$. Then

$$c_d^* \frac{|V(G)|}{A_N} \leq I(G) \leq C_d^* \frac{|V(G)|}{A_N}, \tag{4.2}$$

where the constants c_d^* and C_d^* depend on d only.

Intuitively, we can explain (4.2) as follows. The sides a_m, $m \geq N+1$ are too small relative to the bigger sides, and this means that the box is flat in dimensions $N+1, \ldots, d$ and basically, it has N "real dimensions." In the formula (4.2), $a_{N+1}, \ldots, a_d$ will be on the first power (in $|V(G)|/A_N$), whereas the powers of the first N "real dimensions" $a_1, \ldots, a_N$ will be less than one. The first N dimensions $a_1, \ldots, a_N$ have to be cut by hyperplanes to achieve the integrity bound.

We now show that (4.2) is equivalent to (1.1). By the definition of N, we have $a_{m+1} \geq 2A_m$ for $m = 0, 1, \ldots, N-1$, and if $N < d$, we also have $a_{N+1} < 2A_N$.

Note that $a_{m+1} \geq 2A_m = 2(a_1 \ldots a_m)^{1/(m+1)}$ implies $a_1 \ldots a_{m+1} \geq 2 (a_1 \ldots a_m)^{1+1/(m+1)}$, and so we have $(a_1 \ldots a_{m+1})^{1/(m+2)} \geq b_m (a_1 \cdots a_m)^{1/(m+1)}$, where $b_m = 2^{1/(m+2)} > 1$. Thus $A_{m+1} \geq A_m$ holds for $m = 0, 1, \ldots, N-1$.

Also, $a_{N+1} < 2A_N = 2(a_1 \ldots a_N)^{1/(N+1)}$ implies $a_{N+1}^{1+1/(N+1)} < 2(a_1 \ldots a_{N+1})^{1/(N+1)}$, so $a_{N+2} \leq a_{N+1} < 2(a_1 \ldots a_{N+1})^{1/(N+2)}$. Continuing in this way, we get that $a_{m+1} < 2(a_1 \ldots a_m)^{1/(m+1)}$ holds for $m = N, \ldots, d-1$. As in the previous paragraph, this implies that $A_{m+1} < b_m A_m$ holds for $m = N, \ldots, d-1$.

We conclude that $b_{d-1} b_{d-2} \ldots b_N A_N \geq A_m$ for all $m = 0, \ldots, d$. Here

$$b_{d-1} b_{d-2} \ldots b_N \leq 2^{1/2+1/3+\cdots+1/(d+1)} \leq 2d,$$

and so

$$\frac{|V(G)|}{2d^2 A_N} \leq \frac{|V(G)|}{A_1 + \cdots + A_d} \leq \frac{|V(G)|}{A_N},$$

establishing the equivalence of (4.2) and (1.1). ∎

Proof of Lemma 4.2. First we prove the lower bound in (4.2). We can assume that $1 \leq N$ (otherwise $a_1 < 2A_0$, and so $1 = a_1 = \cdots = a_d$ and G is a single vertex). Then we have $2 \leq a_1$. By definition, when $\mathcal{N} \neq \emptyset$, we have $a_{N+1} < 2A_N$ and $a_N \geq 2A_{N-1}$, whereas when $\mathcal{N} = \emptyset$, we have $a_N \geq 2A_{N-1}$ (and $N = d$). Note that this implies

$$\begin{aligned} a_N &= a_N \frac{(a_1 \ldots a_N)^{\frac{1}{N+1}}}{(a_1 \ldots a_N)^{\frac{1}{N+1}}} = \frac{a_N^{1-\frac{1}{N+1}}}{(a_1 \ldots a_{N-1})^{\frac{1}{N+1}}} A_N \\ &= \frac{a_N^{\frac{N}{N+1}}}{\left((a_1 \ldots a_{N-1})^{\frac{1}{N}}\right)^{\frac{N}{N+1}}} A_N = \left(\frac{a_N}{A_{N-1}}\right)^{\frac{N}{N+1}} A_N \\ &\geq 2^{\frac{N}{N+1}} A_N \geq \sqrt{2} A_N. \end{aligned}$$

So $[\frac{a_i}{A_N}]$ $(i = 1, \ldots, N)$ are positive integers.

Let $k_i \in \{0, \ldots, [\frac{a_i}{A_N}]\}$ $(i = 1, \ldots, N)$ and consider the box-graph

$$\begin{aligned} B(k_1, \ldots, k_N) &:= \Big\{x_1 \in \mathbb{Z} : \ [k_1 A_N] < x_1 \leq \min([(k_1+1)A_N], a_1)\Big\} \\ &\times \ \cdots \ \times \Big\{x_N \in \mathbb{Z} : \ [k_N A_N] < x_N \leq \min([(k_N+1)A_N], a_N)\Big\} \\ &\times \Big\{x_{N+1} \in \mathbb{Z} : \ 0 < x_{N+1} \leq a_{N+1}\Big\} \ \times \ \cdots \ \times \ \Big\{x_d \in \mathbb{Z} : \ 0 < x_d \leq a_d\Big\}, \end{aligned}$$

which, for simplicity, we call a box. When $k_i = [\frac{a_i}{A_N}]$ holds for at least one $i \in \{0, \ldots, N\}$, we call $B(k_1, \ldots, k_N)$ a truncated box (it may even be an empty set), and otherwise we call it a full box. Clearly, the $B(k_1, \ldots, k_N)$ are disjoint sets, and their union is $V(G)$. Since $A_N - 1 < [(k_i+1)A_N] - [k_i A_N] < A_N + 1$, any of the first N dimensions of a full box is an integer in the interval $(A_N - 1, A_N + 1)$. The remaining dimensions of a full box are $a_{N+1}, \ldots, a_d$.

The number of full boxes is $\prod_{i=1}^{N} [\frac{a_i}{A_N}]$. Since $A_N \le a_i/\sqrt{2}$ $(i = 1, \ldots, N)$, we have

$$\begin{aligned} A_N = \prod_{i=1}^{N} \frac{a_i}{A_N} &\ge \prod_{i=1}^{N} \left[\frac{a_i}{A_N}\right] \\ &\ge \prod_{i=1}^{N} \left(\frac{a_i}{A_N} - 1\right) \ge \frac{\prod_{i=1}^{N} \left(a_i - \frac{1}{\sqrt{2}} a_i\right)}{A_N^N} \ge \left(1 - \frac{1}{\sqrt{2}}\right)^d A_N. \end{aligned} \tag{4.3}$$

Let M denote the maximum dimension (i.e., maximal number of vertices on an edge parallel to a coordinate axis) of any of the full boxes. When $\mathcal{N} \neq \emptyset$, using the estimate $2A_N > A_N + 1$ for the first N dimensions and $2A_N > a_{N+1} \ge \cdots \ge a_d$ for the rest of the dimensions, we get $2A_N > M$. When $\mathcal{N} = \emptyset$, using $2A_N > A_N + 1$ (and $N = d$), we get again $2A_N > M$. The minimal number m_d of vertices on any hyperface of an arbitrary full box is at least

$$\begin{aligned} m_d \ge \frac{(A_N - 1)^N a_{N+1} \ldots a_d}{M} &> \frac{(A_N - 1)^N a_{N+1} \ldots a_d}{2A_N} \\ &\ge \frac{1}{2}\left(1 - \frac{1}{\sqrt[d+1]{2}}\right)^d A_N^{N-1} a_{N+1} \ldots a_d, \end{aligned}$$

where we used that the number of vertices in a full box is at least

$$(A_N - 1)^N a_{N+1} \ldots a_d \tag{4.4}$$

and

$$1 - \frac{1}{A_N} \ge 1 - \frac{1}{\sqrt[d+1]{2}}.$$

The last inequality follows from $A_N \ge \sqrt[d+1]{2}$.

Now let $S \subset V(G)$ be arbitrary. Let $0 < \epsilon < 1$ be arbitrary, and let c_d be the number given in the statement of Lemma 4.1.

Case 1. If there exists a full box B such that

$$|S \cap B| \le c_d \frac{1}{2}\left(1 - \frac{1}{\sqrt[d+1]{2}}\right)^d A_N^{N-1} a_{N+1} \ldots a_d \le c_d m_d, \tag{4.5}$$

then, by Lemma 4.1 (and (4.4)), we have

$$I(G) \ge (1-\epsilon)(A_N - 1)^N a_{N+1} \ldots a_d \ge (1-\epsilon)\left(1 - \frac{1}{\sqrt[d+1]{2}}\right)^d A_N^N a_{N+1} \ldots a_d. \tag{4.6}$$

Case 2. If

$$|S \cap B| > c_d \frac{1}{2}\left(1 - \frac{1}{\sqrt[d+1]{2}}\right)^d A_N^{N-1} a_{N+1} \ldots a_d \tag{4.7}$$

for any full box B, then, by (4.3),

$$I(G) \geq |S| > (\text{number of full boxes}) \cdot c_d \frac{1}{2}\left(1 - \frac{1}{\sqrt[d+1]{2}}\right)^d A_N^{N-1} a_{N+1} \dots a_d$$

$$\geq \left(1 - \frac{1}{\sqrt{2}}\right)^d c_d \frac{1}{2}\left(1 - \frac{1}{\sqrt[d+1]{2}}\right)^d A_N^N a_{N+1} \dots a_d. \tag{4.8}$$

Since $A_N^N a_{N+1} \dots a_d = (\prod_{i=1}^d a_i)/A_N = |V(G)|/A_N$, (4.6) and (4.8) establish the lower bound at (4.2).

To prove the upper bound at (4.2), intersect G with hyperplanes to define the boxes in the first half of the proof. More precisely, let

$$H_1 := \left\{[k_1 A_N] :\ k_1 = 1, \dots, \left[\frac{a_1}{A_N}\right]\right\} \times \{1, \dots, a_2\} \times \cdots \times \{1, \dots, a_d\},$$

$$H_2 := \{1, \dots, a_1\} \times \left\{[k_2 A_N] :\ k_2 = 1, \dots, \left[\frac{a_2}{A_N}\right]\right\}$$
$$\times \{1, \dots, a_3\} \times \cdots \times \{1, \dots, a_d\},$$

$$\vdots$$

$$H_N := \{1, \dots, a_1\} \times \cdots \times \{1, \dots, a_{d-1}\} \times \left\{[k_N A_N] :\ k_N = 1, \dots, \left[\frac{a_N}{A_N}\right]\right\},$$

and let $S := \cup_{i=1}^N H_i$. Now,

$$|S| \leq \sum_{i=1}^N |H_i| = \sum_{i=1}^N \left[\frac{a_i}{A_N}\right] \frac{|V(G)|}{a_i} \leq d\frac{|V(G)|}{A_N}.$$

We have seen that any of the first N dimensions of a full box is an integer less than $A_N + 1$, and the next dimensions are $a_{N+1}, \dots, a_d$. Note that this is also true for the truncated boxes (which easily follows from $[([\frac{a_i}{A_N}]+1)A_N] \geq a_i,\ i = 1, \dots, N$). So any box (and hence any component of $G \setminus S$) has at most $(A_N + 1)^N a_{N+1} \dots a_d$ vertices. These lead to

$$I(G) \leq d\frac{|V(G)|}{A_N} + (A_N + 1)^N a_{N+1} \dots a_d$$

$$\leq d\frac{|V(G)|}{A_N} + \left(1 + \frac{1}{\sqrt[d+1]{2}}\right)^d A_N^N a_{N+1} \dots a_d \leq \left(d + \left(1 + \frac{1}{\sqrt[d+1]{2}}\right)^d\right) \frac{|V(G)|}{A_N}, \tag{4.9}$$

where we also used

$$1 + \frac{1}{A_N} \leq 1 + \frac{1}{\sqrt[d+1]{2}}. \qquad \square$$

It is possible to give values to the constants c_d^* and C_d^* in Lemma 4.2. For example, below we consider the case $d = 2$. (Note that one can certainly find better constants for the upper and lower bounds if one considers a proof that applies to the special case $d = 2$ directly.)

Proof of Theorem 1.5. Let $d = 2$. First we give concrete values for the constants in Lemma 4.1. Let $\epsilon \in (0, 1)$. Since in the proof of Lemma 4.1 we have that $c_1 \in (0, 1)$

may be chosen arbitrarily and that C can be as close to one as we wish, $c_2 \in (0,1)$ can be any number satisfying

$$(1-\sqrt{c_2})(1-c_2) > 1-\epsilon.$$

Furthermore, we want to choose ϵ and c_2 to get about the same lower bounds in (4.6) and (4.8). Thus we desire

$$1-\epsilon \approx \frac{1}{2}c_2\left(1-\frac{1}{\sqrt{2}}\right)^2.$$

Set $\epsilon = 0.968$ and $c_2 = 0.754$. Now, (4.6) and (4.8) give $c_2^* = 0.00136$. Together with the upper bound from (4.9), we have

$$0.00136\frac{|V(G)|}{A_N} \leq I(G) \leq 5.22\frac{|V(G)|}{A_N}. \tag{4.10}$$

(This holds even in the case when $N = 0$.)

Now in Lemma 4.2, we have $A_0 = 1$, $A_1 = \sqrt{a_1}$, and $A_2 = \sqrt[3]{a_1a_2}$. Note that if $a_1 < 2A_0 = 2$, then necessarily $a_1 = a_2 = 1$, and the claim of Theorem 1.5 is satisfied. So we may assume that $a_1 \geq 2A_0$.

If $a_2 \geq 2A_1 = 2\sqrt{a_1}$, then $N = 2$ in Lemma 4.2, and so (4.2) leads to

$$0.00136n^{2/3} \leq I(G) \leq 5.22n^{2/3}.$$

If $a_2 < 2A_1 = 2\sqrt{a_1}$, then $N = 1$ in Lemma 4.2, and so (4.2) leads to

$$0.00136\sqrt{a_1}a_2 \leq I(G) \leq 5.22\sqrt{a_1}a_2. \qquad \square$$

Note that the above inequality gives $I(G) = O(\sqrt{n})$ in the degenerate cases when $a_2 = O(1)$. This is consistent with the known integrity of a path ($a_2 = 1$); see [3].

Proof of Theorem 1.6. Let $G \subset \mathbb{Z}^d$ be a cube graph whose dimensions are of size a. Note that the inequality $a_{m+1} < 2A_m$ in Lemma 4.2 now simplifies to $a < 2^{m+1}$. Thus, for $a \geq 2^{m+1}$, we have $\mathcal{N} = \emptyset$, and so $N = d$. Since $|V(G)|/A_d = a^d/a^{d/(d+1)} = a^{d^2/(d+1)} = |V(G)|^{d/(d+1)}$, Lemma 4.2 gives

$$c_d^*|V(G)|^{d/(d+1)} \leq I(G) \leq C_d^*|V(G)|^{d/(d+1)},$$

which holds even in the case $a < 2^{m+1}$ if we redefine the values of the constants c_d^* and C_d^*. $\square$

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